Elementary Linear Algebra

Ron Larson

Seventh Edition

INDEX OF APPLICATIONS

Elementary Linear Algebra

This page intentionally left blank

Elementary Linear Algebra

Seventh Edition

Ron Larson
The Pennsylvania State University
The Behrend College

BROOKS/COLE
CENGAGE Learning®

Australia • Brazil • Japan • Korea • Mexico • Singapore • Spain • United Kingdom • United States

Elementary Linear Algebra
Seventh Edition

Ron Larson

Vice President, Editorial Director: P.J. Boardman
Publisher: Richard Stratton
Senior Sponsoring Editor: Molly Taylor
Senior Development Editor: Laura Wheel
Assistant Editor: Shaylin Walsh Hogan
Editorial Assistant: Alexander Gontar
Associate Media Editor: Andrew Coppola
Senior Marketing Manager: Jennifer Pursley Jones
Marketing Coordinator: Michael Ledesma
Marketing Communications Manager: Mary Anne Payumo
Content Project Manager: Jill Quinn
Manufacturing Planner: Doug Bertke
Rights Acquisition Specialist: Shalice Shah-Caldwell
Text Designer: Larson Texts, Inc.
Cover Designer: Larson Texts, Inc.
Cover Image: © Arnaldo Pomodoro
Compositor: Larson Texts, Inc.

Library of Congress Control Number: 2011935270

Student Edition:
ISBN-13: 978-1-133-11087-3
ISBN-10: 1-133-11087-8

Brooks/Cole
20 Channel Center Street
Boston, MA 02210
USA

Printed in the United States of America
1 2 3 4 5 6 7 15 14 13 12 11

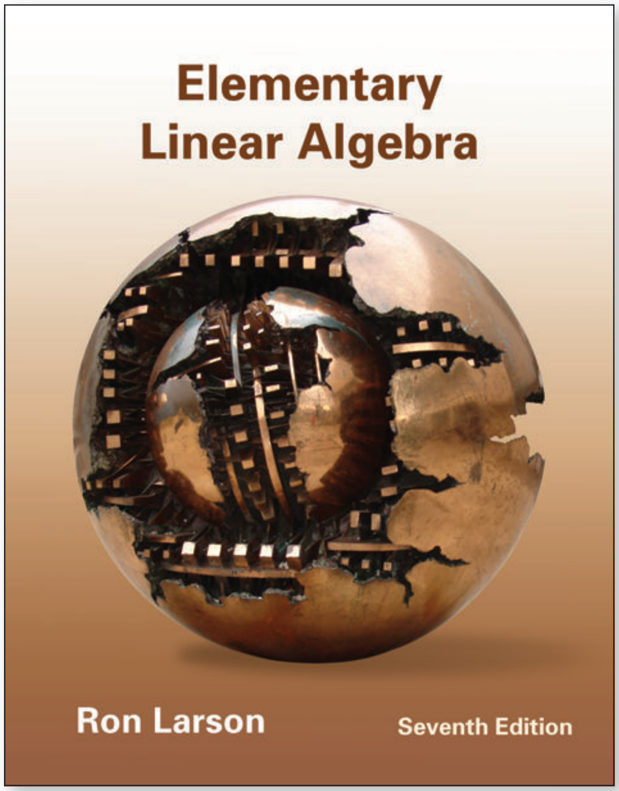
Contents

*Available online at ***www.cengagebrain.com.***

This page intentionally left blank

Preface

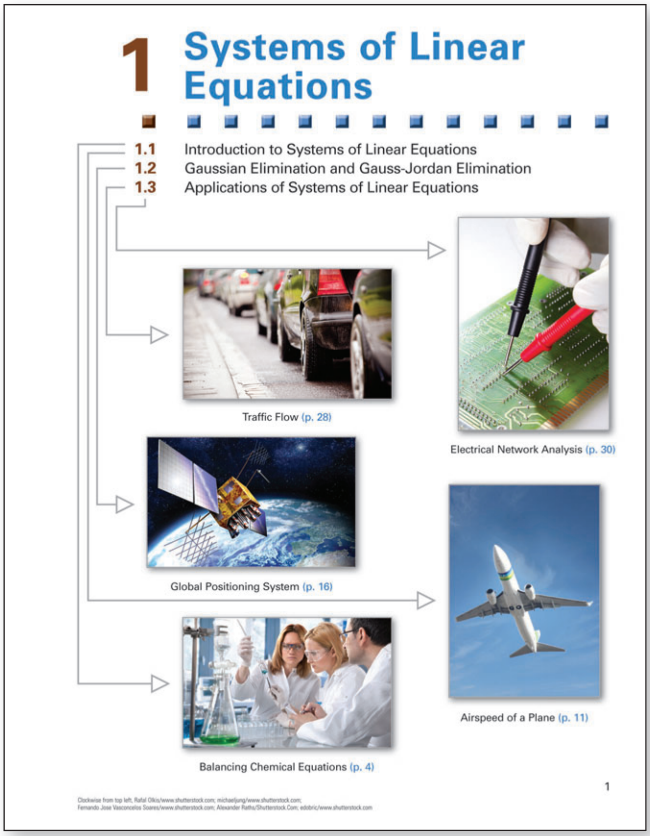
Welcome to the Seventh Edition of *Elementary Linear Algebra*. My primary goal is to present the major concepts of linear algebra clearly and concisely. To this end, I have carefully selected the examples and exercises to balance theory with applications and geometrical intuition. The order and coverage of topics were chosen for maximum efficiency, effectiveness, and balance. The new design complements the multitude of features and applications found throughout the book.



New To This Edition

NEW *Chapter Openers*

Each *Chapter Opener* highlights five real-life applications of linear algebra found throughout the chapter. Many of the applications reference the new *Linear Algebra Applied* featured (discussed below). You can find a full listing of the applications in the *Index of Applications* on the inside front cover.



1 Systems of Linear Equations

1.1 Introduction to Systems of Linear Equations
1.2 Gaussian Elimination and Gauss-Jordan Elimination
1.3 Applications of Systems of Linear Equations

Traffic Flow (p. 28)

Electrical Network Analysis (p. 30)

Global Positioning System (p. 16)

Airspeed of a Plane (p. 11)

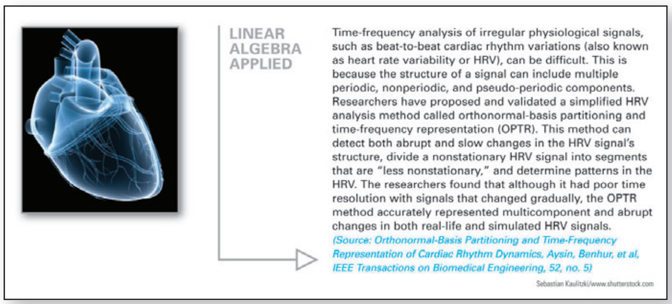
Balancing Chemical Equations (p. 4)

1

Clockwise from top left, Rafal Olkis/www.shutterstock.com; michaeljung/www.shutterstock.com; Fernando Jose Vasconcelos Soares/www.shutterstock.com; Alexander Raths/Shutterstock.Com; edobric/www.shutterstock.com

NEW *Linear Algebra Applied*

Linear Algebra Applied describes a real-life application of concepts discussed in a section. These applications include biology and life sciences, business and economics, engineering and technology, physical sciences, and statistics and probability.



LINEAR ALGEBRA APPLIED

Time-frequency analysis of irregular physiological signals, such as beat-to-beat cardiac rhythm variations (also known as heart rate variability or HRV), can be difficult. This is because the structure of a signal can include multiple periodic, nonperiodic, and pseudo-periodic components. Researchers have proposed and validated a simplified HRV analysis method called orthonormal-basis partitioning and time-frequency representation (OPTR). This method can detect both abrupt and slow changes in the HRV signal's structure, divide a nonstationary HRV signal into segments that are "less nonstationary," and determine patterns in the HRV. The researchers found that although it had poor time resolution with signals that changed gradually, the OPTR method accurately represented multicomponent and abrupt changes in both real-life and simulated HRV signals. *(Source: Orthonormal-Basis Partitioning and Time-Frequency Representation of Cardiac Rhythm Dynamics, Aysin, Benhur, et al, IEEE Transactions on Biomedical Engineering, 52, no. 5)*

Sebastian Kaulitzki/www.shutterstock.com

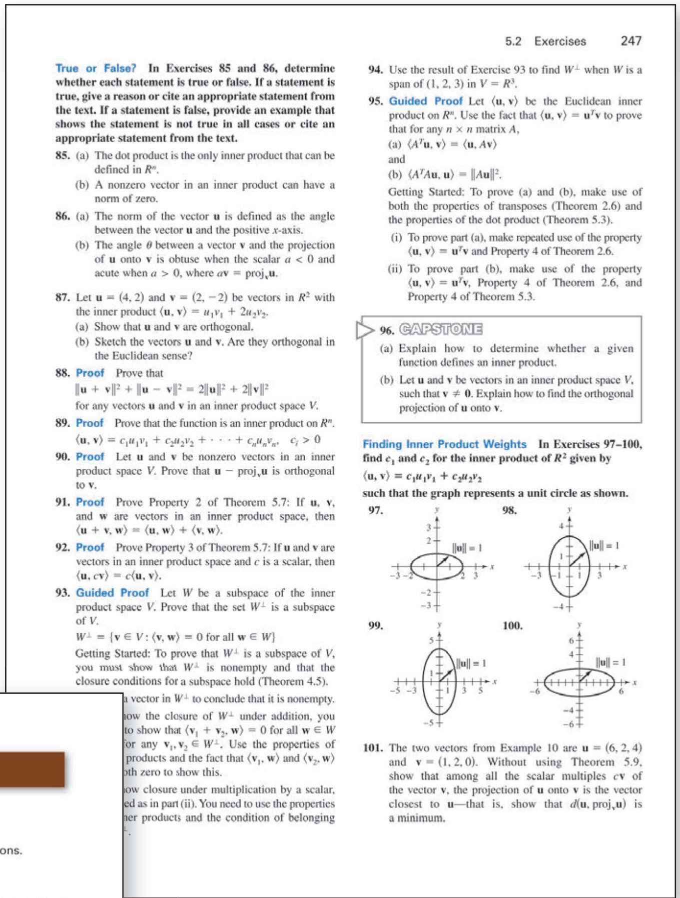
NEW *Capstone Exercises*

The *Capstone* is a conceptual problem that synthesizes key topics to check students' understanding of the section concepts. I recommend it.

REVISED *Exercise Sets*

The exercise sets have been carefully and extensively examined to ensure they are rigorous, relevant, and cover all topics suggested by our users. The exercises have been reorganized and titled so you can better see the connections between examples and exercises. Many new skill building, challenging, and application exercises have been added. As in earlier editions, the following pedagogically-proven types of exercises are included:

- **True or False Exercises** ask students to give examples or justifications to support their conclusions.
- **Proofs**
- **Guided Proofs** lead student through the initial steps of constructing proofs and then utilizing the results.
- **Writing Exercises**
- **Technology Exercises** are indicated throughout the text with .
- Exercises utilizing **electronic data sets** are indicated by and found at *www.cengagebrain.com.*



5.2 Exercises 247

True or False? In Exercises 85 and 86, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

85. (a) The dot product is the only inner product that can be defined in R^n.
(b) A nonzero vector in an inner product can have a norm of zero.

86. (a) The norm of the vector $\mathbf{u}$ is defined as the angle between the vector $\mathbf{u}$ and the positive x-axis.
(b) The angle θ between a vector $\mathbf{v}$ and the projection of $\mathbf{u}$ onto $\mathbf{v}$ is obtuse when the scalar $a < 0$ and acute when $a > 0$, where $a\mathbf{v} = \text{proj}_{\mathbf{v}}\mathbf{u}$.

87. Let $\mathbf{u} = (4, 2)$ and $\mathbf{v} = (2, -2)$ be vectors in R^2 with the inner product $\langle \mathbf{u}, \mathbf{v}\rangle = u_1v_1 + 2u_2v_2$.
(a) Show that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.
(b) Sketch the vectors $\mathbf{u}$ and $\mathbf{v}$. Are they orthogonal in the Euclidean sense?

88. Proof Prove that
$\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$
for any vectors $\mathbf{u}$ and $\mathbf{v}$ in an inner product space V.

89. Proof Prove that the function is an inner product on R^n.
$\langle \mathbf{u}, \mathbf{v}\rangle = c_1u_1v_1 + c_2u_2v_2 + \cdots + c_nu_nv_n, \quad c_i > 0$

90. Proof Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in an inner product space V. Prove that $\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u}$ is orthogonal to $\mathbf{v}$.

91. Proof Prove Property 2 of Theorem 5.7: If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in an inner product space, then $\langle \mathbf{u} + \mathbf{v}, \mathbf{w}\rangle = \langle \mathbf{u}, \mathbf{w}\rangle + \langle \mathbf{v}, \mathbf{w}\rangle$.

92. Proof Prove Property 3 of Theorem 5.7: If $\mathbf{u}$ and $\mathbf{v}$ are vectors in an inner product space and c is a scalar, then $\langle \mathbf{u}, c\mathbf{v}\rangle = c\langle \mathbf{u}, \mathbf{v}\rangle$.

93. Guided Proof Let W be a subspace of the inner product space V. Prove that the set $W^{\perp}$ is a subspace of V.
$W^{\perp} = \{\mathbf{v} \in V : \langle \mathbf{v}, \mathbf{w}\rangle = 0 \text{ for all } \mathbf{w} \in W\}$
Getting Started: To prove that $W^{\perp}$ is a subspace of V, you must show that $W^{\perp}$ is nonempty and that the closure conditions for a subspace hold (Theorem 4.5).
a vector in $W^{\perp}$ to conclude that it is nonempty.
ow the closure of $W^{\perp}$ under addition, you to show that $\langle \mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}\rangle = 0$ for all $\mathbf{w} \in W$ for any $\mathbf{v}_1, \mathbf{v}_2 \in W^{\perp}$. Use the properties of products and the fact that $\langle \mathbf{v}_1, \mathbf{w}\rangle$ and $\langle \mathbf{v}_2, \mathbf{w}\rangle$ oth zero to show this.
ow closure under multiplication by a scalar, ed as in part (ii). You need to use the properties er products and the condition of belonging

94. Use the result of Exercise 93 to find $W^{\perp}$ when W is a span of (1, 2, 3) in $V = R^3$.

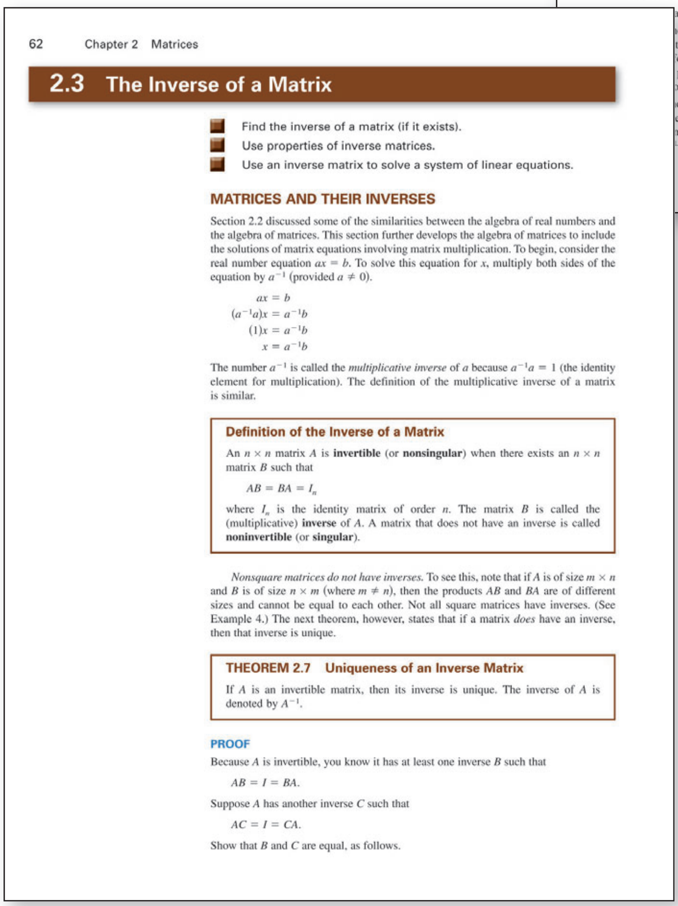
95. Guided Proof Let $\langle \mathbf{u}, \mathbf{v}\rangle$ be the Euclidean inner product on R^n. Use the fact that $\langle \mathbf{u}, \mathbf{v}\rangle = \mathbf{u}^T\mathbf{v}$ to prove that for any $n \times n$ matrix A,
(a) $\langle A^T\mathbf{u}, \mathbf{v}\rangle = \langle \mathbf{u}, A\mathbf{v}\rangle$
and
(b) $\langle A^TA\mathbf{u}, \mathbf{u}\rangle = \|A\mathbf{u}\|^2$.
Getting Started: To prove (a) and (b), make use of both the properties of transposes (Theorem 2.6) and the properties of the dot product (Theorem 5.3).
(i) To prove part (a), make repeated use of the property $\langle \mathbf{u}, \mathbf{v}\rangle = \mathbf{u}^T\mathbf{v}$ and Property 4 of Theorem 2.6.
(ii) To prove part (b), make use of the property $\langle \mathbf{u}, \mathbf{v}\rangle = \mathbf{u}^T\mathbf{v}$, Property 4 of Theorem 2.6, and Property 4 of Theorem 5.3.

96. CAPSTONE
(a) Explain how to determine whether a given function defines an inner product.
(b) Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in an inner product space V, such that $\mathbf{v} \neq \mathbf{0}$. Explain how to find the orthogonal projection of $\mathbf{u}$ onto $\mathbf{v}$.

Finding Inner Product Weights In Exercises 97–100, find c_1 and c_2 for the inner product of R^2 given by
$\langle \mathbf{u}, \mathbf{v}\rangle = c_1u_1v_1 + c_2u_2v_2$
such that the graph represents a unit circle as shown.

97. **98.** **99.** **100.**

101. The two vectors from Example 10 are $\mathbf{u} = (6, 2, 4)$ and $\mathbf{v} = (1, 2, 0)$. Without using Theorem 5.9, show that among all the scalar multiples $c\mathbf{v}$ of the vector $\mathbf{v}$, the projection of $\mathbf{u}$ onto $\mathbf{v}$ is the vector closest to $\mathbf{u}$—that is, show that $d(\mathbf{u}, \text{proj}_{\mathbf{v}}\mathbf{u})$ is a minimum.



62 Chapter 2 Matrices

2.3 The Inverse of a Matrix

- Find the inverse of a matrix (if it exists).
- Use properties of inverse matrices.
- Use an inverse matrix to solve a system of linear equations.

MATRICES AND THEIR INVERSES

Section 2.2 discussed some of the similarities between the algebra of real numbers and the algebra of matrices. This section further develops the algebra of matrices to include the solutions of matrix equations involving matrix multiplication. To begin, consider the real number equation $ax = b$. To solve this equation for x, multiply both sides of the equation by a^{-1} (provided $a \neq 0$).

$$ax = b$$
$$(a^{-1}a)x = a^{-1}b$$
$$(1)x = a^{-1}b$$
$$x = a^{-1}b$$

The number a^{-1} is called the *multiplicative inverse* of a because $a^{-1}a = 1$ (the identity element for multiplication). The definition of the multiplicative inverse of a matrix is similar.

Definition of the Inverse of a Matrix

An $n \times n$ matrix A is **invertible** (or **nonsingular**) when there exists an $n \times n$ matrix B such that

$$AB = BA = I_n$$

where I_n is the identity matrix of order n. The matrix B is called the (multiplicative) **inverse** of A. A matrix that does not have an inverse is called **noninvertible** (or **singular**).

Nonsquare matrices do not have inverses. To see this, note that if A is of size $m \times n$ and B is of size $n \times m$ (where $m \neq n$), then the products AB and BA are of different sizes and cannot be equal to each other. Not all square matrices have inverses. (See Example 4.) The next theorem, however, states that if a matrix *does* have an inverse, then that inverse is unique.

THEOREM 2.7 Uniqueness of an Inverse Matrix

If A is an invertible matrix, then its inverse is unique. The inverse of A is denoted by A^{-1}.

PROOF

Because A is invertible, you know it has at least one inverse B such that

$$AB = I = BA.$$

Suppose A has another inverse C such that

$$AC = I = CA.$$

Show that B and C are equal, as follows.

Table of Contents Changes

Based on feedback from users, Section 3.4 (Introduction to Eigenvalues) in the previous edition has been removed and its content has been absorbed in Chapter 7 (Eigenvalues and Eigenvectors).

Trusted Features

Section Objectives

A bulleted list of learning objectives provides you the opportunity to preview what will be presented in the upcoming section. For the Seventh Edition, the section objectives are located by relevance at the beginning of each section.

Theorems, Definitions, and Properties

Presented in clear and mathematically precise language, all theorems, definitions, and properties are highlighted for emphasis and easy reference.

Proofs in Outline Form

In addition to proofs in the exercises, some proofs are presented in outline form, omitting the need for burdensome calculations.

Discovery

Discovery helps you develop an intuitive understanding of mathematical concepts and relationships.

EXAMPLE 4 **Finding a Transition Matrix**

Find the transition matrix from B to B' for the following bases for R^3.

$$B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\} \quad \text{and} \quad B' = \{(1, 0, 1), (0, -1, 2), (2, 3, -5)\}$$

SOLUTION

First use the vectors in the two bases to form the matrices B and B'.

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B' = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 1 & 2 & -5 \end{bmatrix}$$

Then form the matrix $[B' \; B]$ and use Gauss-Jordan elimination to rewrite $[B' \; B]$ as $[I_3 \; P^{-1}]$.

$$\left[\begin{array}{rrrrrr} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 1 & 0 \\ 1 & 2 & -5 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{rrrrrr} 1 & 0 & 0 & -1 & 4 & 2 \\ 0 & 1 & 0 & 3 & -7 & -3 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{array}\right]$$

From this, you can conclude that the transition matrix from B to B' is

$$P^{-1} = \begin{bmatrix} -1 & 4 & 2 \\ 3 & -7 & -3 \\ 1 & -2 & -1 \end{bmatrix}.$$

Try multiplying P^{-1} by the coordinate matrix of $\mathbf{x} = [1 \;\; 2 \;\; -1]^T$ to see that the result is the same as the one obtained in Example 3.

DISCOVERY

1. Let $B = \{(1, 0), (1, 2)\}$ and $B' = \{(1, 0), (0, 1)\}$. Form the matrix $[B' \; B]$.
2. Make a conjecture about the necessity of using Gauss-Jordan elimination to obtain the transition matrix P^{-1} when the change of basis is from a nonstandard basis to a standard basis.

TECHNOLOGY

Many graphing utilities and software programs can calculate the determinant of a square matrix. If you use a graphing utility, then you may see something similar to the following for Example 4. **The Online Technology Guide,** available at *www.cengagebrain.com,* provides syntax for programs applicable to Example 4.

```
A
       [[1  -2 3 0 ]
        [-1  1 0 2 ]
        [0   2 0 3 ]
        [3   4 0 -2]]
det A
                 39
```

SOLUTION

After inspecting ... zeros. So, to eli...

$$|A| = 3(C_{13}) + 0(C_{23}) + 0(C_{33}) + 0(C_{43})$$

Because C_{23}, C_{33}, and C_{43} have zero coefficients, you need only find the cofactor C_{13}. To do this, delete the first row and third column of A and evaluate the determinant of the resulting matrix.

$$C_{13} = (-1)^{1+3}\begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{vmatrix} \qquad \text{Delete 1st row and 3rd column.}$$

$$= \begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{vmatrix} \qquad \text{Simplify.}$$

Expanding by cofactors in the second row yields

$$C_{13} = (0)(-1)^{2+1}\begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix} + (2)(-1)^{2+2}\begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix} + (3)(-1)^{2+3}\begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix}$$

$$= 0 + 2(1)(-4) + 3(-1)(-7)$$

$$= 13.$$

You obtain

$$|A| = 3(13)$$

$$= 39.$$

Technology Notes

Technology notes show how you can use graphing utilities and software programs appropriately in the problem-solving process. Many of the technology notes reference the **Online Technology Guide,** located at *www.cengagebrain.com*.

Chapter Projects

Two per chapter, these offer the opportunity for group activities or more extensive homework assignments, and are focused on theoretical concepts or applications. Many encourage the use of technology.

102 Chapter 2 Matrices

2 Projects

1 Exploring Matrix Multiplication

	Test 1	*Test 2*
Anna	84	96
Bruce	56	72
Chris	78	83
David	82	91

The table shows the first two test scores for Anna, Bruce, Chris, and David. Use the table to create a matrix M to represent the data. Input M into a software program or a graphing utility and use it to answer the following questions.

1. Which test was more difficult? Which was easier? Explain.
2. How would you rank the performances of the four students?
3. Describe the meanings of the matrix products $M\begin{bmatrix}1\\0\end{bmatrix}$ and $M\begin{bmatrix}0\\1\end{bmatrix}$.
4. Describe the meanings of the matrix products $[1 \;\; 0 \;\; 0 \;\; 0]M$ and $[0 \;\; 0 \;\; 1 \;\; 0]M$.
5. Describe the meanings of the matrix products $M\begin{bmatrix}1\\1\end{bmatrix}$ and $\frac{1}{2}M\begin{bmatrix}1\\1\end{bmatrix}$.
6. Describe the meanings of the matrix products $[1 \;\; 1 \;\; 1 \;\; 1]M$ and $\frac{1}{4}[1 \;\; 1 \;\; 1 \;\; 1]M$.
7. Describe the meaning of the matrix product $[1 \;\; 1 \;\; 1 \;\; 1]M\begin{bmatrix}1\\1\end{bmatrix}$.
8. Use matrix multiplication to find the combined overall average score on both tests.
9. How could you use matrix multiplication to scale the scores on test 1 by a factor of 1.1?

2 Nilpotent Matrices

Let A be a nonzero square matrix. Is it possible that a positive integer k exists such that $A^k = O$? For example, find A^3 for the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

A square matrix A is **nilpotent of index** k when $A \neq O, A^2 \neq O, \ldots, A^{k-1} \neq O$, but $A^k = O$. In this project you will explore nilpotent matrices.

1. The matrix in the example given above is nilpotent. What is its index?
2. Use a software program or a graphing utility to determine which of the following matrices are nilpotent and find their indices.

(a) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (f) $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

3. Find 3×3 nilpotent matrices of indices 2 and 3.
4. Find 4×4 nilpotent matrices of indices 2, 3, and 4.
5. Find a nilpotent matrix of index 5.
6. Are nilpotent matrices invertible? Prove your answer.
7. When A is nilpotent, what can you say about A^T? Prove your answer.
8. Show that if A is nilpotent, then $I - A$ is invertible.

Supri Suharjoto/Shutterstock.com

Instructor Resources

Print

Instructor's Solutions Manual
ISBN-13: 978-1-133-36538-9
The *Instructor's Solutions Manual* provides worked-out solutions for all even-numbered exercises in the text.

Media

Solution Builder
www.cengage.com/solutionbuilder
This online instructor database offers complete worked-out solutions of all exercises in the text. Solution Builder allows you to create customized, secure solutions printouts (in PDF format) matched exactly to the problems you assign in class.

Diploma Computerized Testing
ISBN-13: 978-1-133-49045-6
Diploma is an easy-to-use assessment software containing hundreds of algorithmic questions derived from the text exercises. With Diploma, you can quickly create, customize, and deliver tests in both print and online formats.

ENHANCED WebAssign
www.webassign.net
WebAssign's homework delivery system lets you deliver, collect, grade, and record assignments via the web. Enhanced WebAssign includes Cengage YouBook—a customizable, media-rich interactive eBook, Personal Study Plans, a Show My Work feature, quizzes, videos, and more!

Cengage YouBook
Cengage YouBook is an interactive and customizable eBook! Containing all the content from the printed text, YouBook features a text edit tool that allows you to modify the textbook narrative as needed. With YouBook, you can quickly re-order entire sections and chapters or hide any content not covered in the course to create an eBook that perfectly matches your syllabus. You can further customize the text by publishing web links or adding YouTube™ videos. Additional media assets include: animated figures, video clips, highlighting, notes, and more! YouBook is available in Enhanced WebAssign.

Student Resources

Print

Student Solutions Manual

ISBN-13: 978-1-133-11132-0

The *Student Solutions Manual* provides complete worked-out solutions to all odd-numbered exercises in the text. Also included are the solutions to all Cumulative Test problems.

Media

ENHANCED WebAssign

www.webassign.net

Enhanced WebAssign is an online homework system that lets instructors deliver, collect, grade, and record assignments via the web. Enhanced WebAssign includes Cengage YouBook—a media-rich interactive ebook, Personal Study Plans, a Show My Work feature, quizzes, videos, and more! Be sure to check with your instructor to find out if Enhanced WebAssign is required for your course.

CengageBrain.com

To access additional course materials and companion resources, please visit *www.cengagebrain.com*. At the CengageBrain.com home page, search for the ISBN of your title (from the back cover of your book) using the search box at the top of the page. This will take you to the product page where free companion resources can be found.

Acknowledgements

I would like to thank the many people who have helped me during various stages of writing this new edition. In particular, I appreciate the efforts of the following colleagues who made many helpful suggestions along the way:

Michael Brown, *San Diego Mesa College*

Nasser Dastrange, *Buena Vista University*

Mike Daven, *Mount Saint Mary College*

David Hemmer, *University of Buffalo, SUNY*

Wai Lau, *Seattle Pacific University*

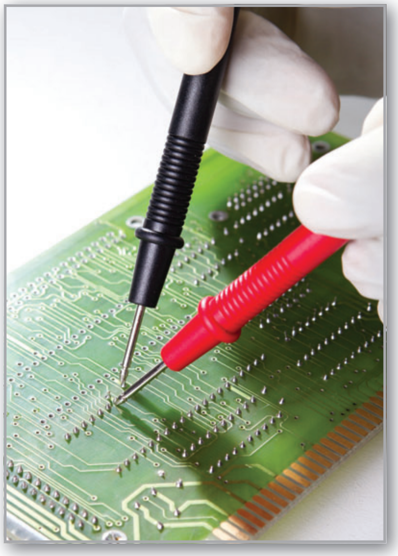
Jorge Sarmiento, *County College of Morris*

I would like to thank Bruce H. Edwards, University of Florida, and David C. Falvo, The Pennsylvania State University, The Behrend College, for their contributions to previous editions of *Elementary Linear Algebra*.

On a personal level, I am grateful to my spouse, Deanna Gilbert Larson, for her love, patience, and support. Also, a special thanks goes to R. Scott O'Neil.

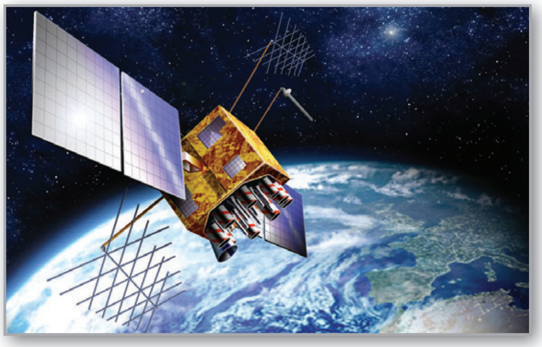
Ron Larson, Ph.D.
Professor of Mathematics
Penn State University
www.RonLarson.com

1 Systems of Linear Equations



Electrical Network Analysis (p. 30)

Traffic Flow (p. 28)



Global Positioning System (p. 16)

Airspeed of a Plane (p. 11)

Balancing Chemical Equations (p. 4)

Clockwise from top left, Rafal Olkis/www.shutterstock.com; michaeljung/www.shutterstock.com;
Fernando Jose Vasconcelos Soares/www.shutterstock.com; Alexander Raths/Shutterstock.Com; edobric/www.shutterstock.com

1.1 Introduction to Systems of Linear Equations

- Recognize a linear equation in *n* variables.
- Find a parametric representation of a solution set.
- Determine whether a system of linear equations is consistent or inconsistent.
- Use back-substitution and Gaussian elimination to solve a system of linear equations.

LINEAR EQUATIONS IN *n* VARIABLES

The study of linear algebra demands familiarity with algebra, analytic geometry, and trigonometry. Occasionally, you will find examples and exercises requiring a knowledge of calculus; these are clearly marked in the text.

Early in your study of linear algebra, you will discover that many of the solution methods involve multiple arithmetic steps, so it is essential to check your work. Use a computer or calculator to check your work and perform routine computations.

Although you will be familiar with some material in this chapter, you should carefully study the methods presented in this chapter. This will cultivate and clarify your intuition for the more abstract material that follows.

Recall from analytic geometry that the equation of a line in two-dimensional space has the form

$$a_1x + a_2y = b, \quad a_1, a_2, \text{ and } b \text{ are constants.}$$

This is a **linear equation in two variables** x and y. Similarly, the equation of a plane in three-dimensional space has the form

$$a_1x + a_2y + a_3z = b, \quad a_1, a_2, a_3, \text{ and } b \text{ are constants.}$$

This is a **linear equation in three variables** x, y, and z. In general, a linear equation in n variables is defined as follows.

REMARK

Letters that occur early in the alphabet are used to represent constants, and letters that occur late in the alphabet are used to represent variables.

Definition of a Linear Equation in *n* Variables

A **linear equation in *n* variables** $x_1, x_2, x_3, \ldots, x_n$ has the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b.$$

The **coefficients** $a_1, a_2, a_3, \ldots, a_n$ are real numbers, and the **constant term** b is a real number. The number a_1 is the **leading coefficient,** and x_1 is the **leading variable.**

Linear equations have no products or roots of variables and no variables involved in trigonometric, exponential, or logarithmic functions. Variables appear only to the first power.

EXAMPLE 1 **Linear and Nonlinear Equations**

Each equation is linear.

a. $3x + 2y = 7$ **b.** $\frac{1}{2}x + y - \pi z = \sqrt{2}$ **c.** $(\sin \pi)x_1 - 4x_2 = e^2$

Each equation is not linear.

a. $xy + z = 2$ **b.** $e^x - 2y = 4$ **c.** $\sin x_1 + 2x_2 - 3x_3 = 0$

SOLUTIONS AND SOLUTION SETS

A **solution** of a linear equation in n variables is a sequence of n real numbers $s_1, s_2, s_3, \ldots, s_n$ arranged to satisfy the equation when you substitute the values

$$x_1 = s_1, \quad x_2 = s_2, \quad x_3 = s_3, \quad \ldots, \quad x_n = s_n$$

into the equation. For example, $x_1 = 2$ and $x_2 = 1$ satisfy the equation $x_1 + 2x_2 = 4$. Some other solutions are $x_1 = -4$ and $x_2 = 4$, $x_1 = 0$ and $x_2 = 2$, and $x_1 = -2$ and $x_2 = 3$.

The set of *all* solutions of a linear equation is called its **solution set,** and when you have found this set, you have **solved** the equation. To describe the entire solution set of a linear equation, use a **parametric representation,** as illustrated in Examples 2 and 3.

EXAMPLE 2 Parametric Representation of a Solution Set

Solve the linear equation $x_1 + 2x_2 = 4$.

SOLUTION

To find the solution set of an equation involving two variables, solve for one of the variables in terms of the other variable. Solving for x_1 in terms of x_2, you obtain

$$x_1 = 4 - 2x_2.$$

In this form, the variable x_2 is **free,** which means that it can take on any real value. The variable x_1 is not free because its value depends on the value assigned to x_2. To represent the infinitely many solutions of this equation, it is convenient to introduce a third variable t called a **parameter.** By letting $x_2 = t$, you can represent the solution set as

$$x_1 = 4 - 2t, \quad x_2 = t, \quad t \text{ is any real number.}$$

To obtain particular solutions, assign values to the parameter t. For instance, $t = 1$ yields the solution $x_1 = 2$ and $x_2 = 1$, and $t = 4$ yields the solution $x_1 = -4$ and $x_2 = 4$.

To parametrically represent the solution set of the linear equation in Example 2 another way, you could have chosen x_1 to be the free variable. The parametric representation of the solution set would then have taken the form

$$x_1 = s, \quad x_2 = 2 - \tfrac{1}{2}s, \quad s \text{ is any real number.}$$

For convenience, choose the variables that occur last in a given equation to be free variables.

EXAMPLE 3 Parametric Representation of a Solution Set

Solve the linear equation $3x + 2y - z = 3$.

SOLUTION

Choosing y and z to be the free variables, solve for x to obtain

$$\begin{aligned} 3x &= 3 - 2y + z \\ x &= 1 - \tfrac{2}{3}y + \tfrac{1}{3}z. \end{aligned}$$

Letting $y = s$ and $z = t$, you obtain the parametric representation

$$x = 1 - \tfrac{2}{3}s + \tfrac{1}{3}t, \quad y = s, \quad z = t$$

where s and t are any real numbers. Two particular solutions are

$$x = 1, y = 0, z = 0 \quad \text{and} \quad x = 1, y = 1, z = 2.$$

SYSTEMS OF LINEAR EQUATIONS

A **system of *m* linear equations in *n* variables** is a set of m equations, each of which is linear in the same n variables:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

REMARK

The double-subscript notation indicates a_{ij} is the coefficient of x_j in the ith equation.

A **solution** of a system of linear equations is a sequence of numbers $s_1, s_2, s_3, \ldots, s_n$ that is a solution of each of the linear equations in the system. For example, the system

$$\begin{aligned} 3x_1 + 2x_2 &= 3 \\ -x_1 + x_2 &= 4 \end{aligned}$$

has $x_1 = -1$ and $x_2 = 3$ as a solution because $x_1 = -1$ and $x_2 = 3$ satisfy *both* equations. On the other hand, $x_1 = 1$ and $x_2 = 0$ is not a solution of the system because these values satisfy only the first equation in the system.

DISCOVERY

1. Graph the two lines

$$\begin{aligned} 3x - y &= 1 \\ 2x - y &= 0 \end{aligned}$$

in the xy-plane. Where do they intersect? How many solutions does this system of linear equations have?

2. Repeat this analysis for the pairs of lines

$$\begin{aligned} 3x - y &= 1 \\ 3x - y &= 0 \end{aligned}$$

and

$$\begin{aligned} 3x - y &= 1 \\ 6x - 2y &= 2. \end{aligned}$$

3. What basic types of solution sets are possible for a system of two equations in two unknowns?

LINEAR ALGEBRA APPLIED

In a chemical reaction, atoms reorganize in one or more substances. For instance, when methane gas (CH_4) combines with oxygen (O_2) and burns, carbon dioxide (CO_2) and water (H_2O) form. Chemists represent this process by a chemical equation of the form

$$(x_1)CH_4 + (x_2)O_2 \rightarrow (x_3)CO_2 + (x_4)H_2O.$$

Because a chemical reaction can neither create nor destroy atoms, all of the atoms represented on the left side of the arrow must be accounted for on the right side of the arrow. This is called *balancing* the chemical equation. In the given example, chemists can use a system of linear equations to find values of x_1, x_2, x_3, and x_4 that will balance the chemical equation.

It is possible for a system of linear equations to have exactly one solution, infinitely many solutions, or no solution. A system of linear equations is **consistent** when it has at least one solution and **inconsistent** when it has no solution.

EXAMPLE 4 Systems of Two Equations in Two Variables

Solve and graph each system of linear equations.

a. $\begin{aligned} x + y &= 3 \\ x - y &= -1 \end{aligned}$ **b.** $\begin{aligned} x + y &= 3 \\ 2x + 2y &= 6 \end{aligned}$ **c.** $\begin{aligned} x + y &= 3 \\ x + y &= 1 \end{aligned}$

SOLUTION

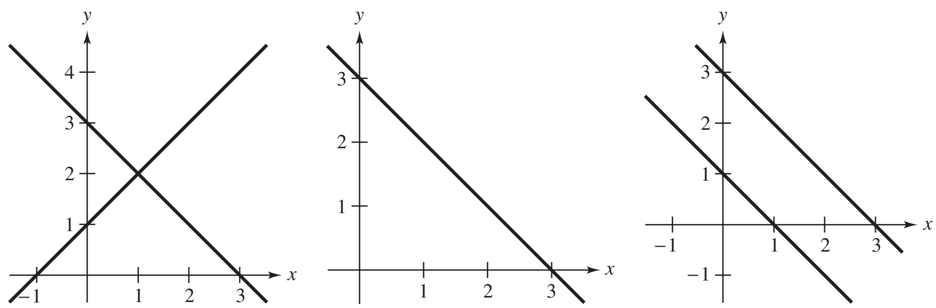
a. This system has exactly one solution, $x = 1$ and $y = 2$. One way to obtain the solution is to add the two equations to give $2x = 2$, which implies $x = 1$ and so $y = 2$. The graph of this system is two *intersecting* lines, as shown in Figure 1.1(a).

b. This system has infinitely many solutions because the second equation is the result of multiplying both sides of the first equation by 2. A parametric representation of the solution set is

$$x = 3 - t, \quad y = t, \quad t \text{ is any real number.}$$

The graph of this system is two *coincident* lines, as shown in Figure 1.1(b).

c. This system has no solution because the sum of two numbers cannot be 3 and 1 simultaneously. The graph of this system is two *parallel* lines, as shown in Figure 1.1(c).



a. Two intersecting lines: $\begin{aligned} x + y &= 3 \\ x - y &= -1 \end{aligned}$

b. Two coincident lines: $\begin{aligned} x + y &= 3 \\ 2x + 2y &= 6 \end{aligned}$

c. Two parallel lines: $\begin{aligned} x + y &= 3 \\ x + y &= 1 \end{aligned}$

Figure 1.1

Example 4 illustrates the three basic types of solution sets that are possible for a system of linear equations. This result is stated here without proof. (The proof is provided later in Theorem 2.5.)

Number of Solutions of a System of Linear Equations

For a system of linear equations, precisely one of the following is true.

1. The system has exactly one solution (consistent system).
2. The system has infinitely many solutions (consistent system).
3. The system has no solution (inconsistent system).

SOLVING A SYSTEM OF LINEAR EQUATIONS

Which system is easier to solve algebraically?

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y \qquad &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned} \qquad\qquad \begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ z &= 2 \end{aligned}$$

The system on the right is clearly easier to solve. This system is in **row-echelon form,** which means that it has a "stair-step" pattern with leading coefficients of 1. To solve such a system, use a procedure called **back-substitution.**

EXAMPLE 5 Using Back-Substitution in Row-Echelon Form

Use back-substitution to solve the system.

$$\begin{aligned} x - 2y &= 5 && \text{Equation 1} \\ y &= -2 && \text{Equation 2} \end{aligned}$$

SOLUTION

From Equation 2, you know that $y = -2$. By substituting this value of y into Equation 1, you obtain

$$\begin{aligned} x - 2(-2) &= 5 && \text{Substitute } -2 \text{ for } y. \\ x &= 1. && \text{Solve for } x. \end{aligned}$$

The system has exactly one solution: $x = 1$ and $y = -2$.

The term *back-substitution* implies that you work *backwards*. For instance, in Example 5, the second equation gives you the value of y. Then you substitute that value into the first equation to solve for x. Example 6 further demonstrates this procedure.

EXAMPLE 6 Using Back-Substitution in Row-Echelon Form

Solve the system.

$$\begin{aligned} x - 2y + 3z &= 9 && \text{Equation 1} \\ y + 3z &= 5 && \text{Equation 2} \\ z &= 2 && \text{Equation 3} \end{aligned}$$

SOLUTION

From Equation 3, you know the value of z. To solve for y, substitute $z = 2$ into Equation 2 to obtain

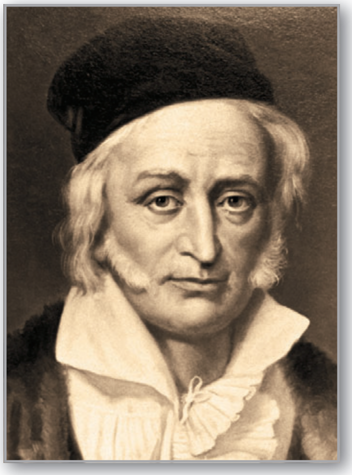
$$\begin{aligned} y + 3(2) &= 5 && \text{Substitute 2 for } z. \\ y &= -1. && \text{Solve for } y. \end{aligned}$$

Then, substitute $y = -1$ and $z = 2$ in Equation 1 to obtain

$$\begin{aligned} x - 2(-1) + 3(2) &= 9 && \text{Substitute } -1 \text{ for } y \text{ and 2 for } z. \\ x &= 1. && \text{Solve for } x. \end{aligned}$$

The solution is $x = 1$, $y = -1$, and $z = 2$.

Two systems of linear equations are **equivalent** when they have the same solution set. To solve a system that is not in row-echelon form, first convert it to an *equivalent* system that is in row-echelon form by using the operations listed on the next page.



Carl Friedrich Gauss (1777–1855)

German mathematician Carl Friedrich Gauss is recognized, with Newton and Archimedes, as one of the three greatest mathematicians in history. Gauss used a form of what is now known as Gaussian elimination in his research. Although this method was named in his honor, the Chinese used an almost identical method some 2000 years prior to Gauss.

Operations That Produce Equivalent Systems

Each of the following operations on a system of linear equations produces an *equivalent* system.

1. Interchange two equations.
2. Multiply an equation by a nonzero constant.
3. Add a multiple of an equation to another equation.

Rewriting a system of linear equations in row-echelon form usually involves a *chain* of equivalent systems, each of which is obtained by using one of the three basic operations. This process is called **Gaussian elimination,** after the German mathematician Carl Friedrich Gauss (1777–1855).

EXAMPLE 7 Using Elimination to Rewrite a System in Row-Echelon Form

Solve the system.

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

SOLUTION

Although there are several ways to begin, you want to use a systematic procedure that is easily applicable to large systems. Work from the upper left corner of the system, saving the x at the upper left and eliminating the other x-terms from the first column.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

Adding the first equation to the second equation produces a new second equation.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ -y - z &= -1 \end{aligned}$$

Adding -2 times the first equation to the third equation produces a new third equation.

Now that you have eliminated all but the first x from the first column, work on the second column.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2z &= 4 \end{aligned}$$

Adding the second equation to the third equation produces a new third equation.

$$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ z &= 2 \end{aligned}$$

Multiplying the third equation by $\frac{1}{2}$ produces a new third equation.

This is the same system you solved in Example 6, and, as in that example, the solution is

$$x = 1, \quad y = -1, \quad z = 2.$$

Each of the three equations in Example 7 represents a plane in a three-dimensional coordinate system. Because the unique solution of the system is the point

$$(x, y, z) = (1, -1, 2)$$

the three planes intersect at this point, as shown in Figure 1.2.

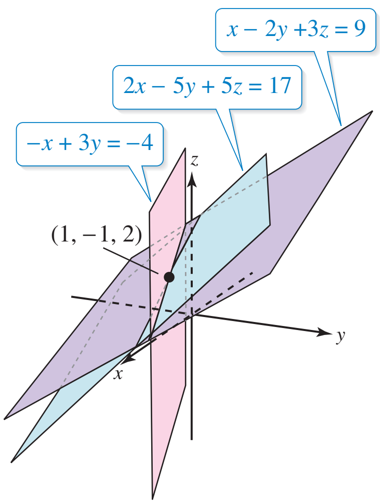


Figure 1.2

Bettmann/Corbis

Because many steps are required to solve a system of linear equations, it is very easy to make arithmetic errors. So, you should develop the habit of *checking your solution by substituting it into each equation in the original system.* For instance, in Example 7, you can check the solution $x = 1$, $y = -1$, and $z = 2$ as follows.

$$\begin{aligned} &\text{Equation 1:} \quad (1) - 2(-1) + 3(2) = 9 \\ &\text{Equation 2:} \quad -(1) + 3(-1) = -4 \\ &\text{Equation 3:} \quad 2(1) - 5(-1) + 5(2) = 17 \end{aligned}$$

Substitute solution in each equation of the original system.

The next example involves an inconsistent system—one that has no solution. The key to recognizing an inconsistent system is that at some stage of the elimination process, you obtain a false statement such as $0 = -2$.

EXAMPLE 8 An Inconsistent System

Solve the system.

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 1 \\ 2x_1 - x_2 - 2x_3 &= 2 \\ x_1 + 2x_2 - 3x_3 &= -1 \end{aligned}$$

SOLUTION

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 1 \\ 5x_2 - 4x_3 &= 0 \\ x_1 + 2x_2 - 3x_3 &= -1 \end{aligned}$$

Adding -2 times the first equation to the second equation produces a new second equation.

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 1 \\ 5x_2 - 4x_3 &= 0 \\ 5x_2 - 4x_3 &= -2 \end{aligned}$$

Adding -1 times the first equation to the third equation produces a new third equation.

(Another way of describing this operation is to say that you *subtracted* the first equation from the third equation to produce a new third equation.)

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 1 \\ 5x_2 - 4x_3 &= 0 \\ 0 &= -2 \end{aligned}$$

Adding -1 times the second equation to the third equation produces a new third equation.

Because $0 = -2$ is a false statement, this system has no solution. Moreover, because this system is equivalent to the original system, the original system also has no solution.

As in Example 7, the three equations in Example 8 represent planes in a three-dimensional coordinate system. In this example, however, the system is inconsistent. So, the planes do not have a point in common, as shown in Figure 1.3.

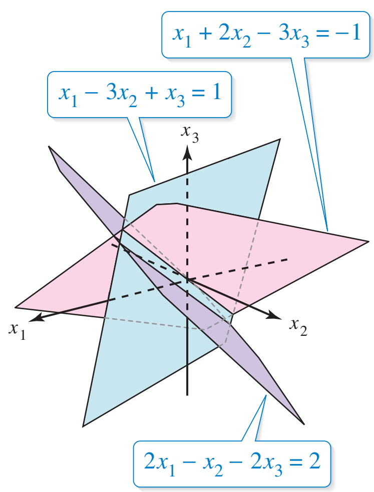


Figure 1.3

This section ends with an example of a system of linear equations that has infinitely many solutions. You can represent the solution set for such a system in parametric form, as you did in Examples 2 and 3.

EXAMPLE 9 A System with Infinitely Many Solutions

Solve the system.

$$\begin{aligned} x_2 - x_3 &= 0 \\ x_1 \qquad - 3x_3 &= -1 \\ -x_1 + 3x_2 \qquad &= 1 \end{aligned}$$

SOLUTION

Begin by rewriting the system in row-echelon form, as follows.

$$\begin{aligned} x_1 \qquad - 3x_3 &= -1 \\ x_2 - x_3 &= 0 \\ -x_1 + 3x_2 \qquad &= 1 \end{aligned}$$

Interchange the first two equations.

$$\begin{aligned} x_1 \qquad - 3x_3 &= -1 \\ x_2 - x_3 &= 0 \\ 3x_2 - 3x_3 &= 0 \end{aligned}$$

Adding the first equation to the third equation produces a new third equation.

$$\begin{aligned} x_1 \qquad - 3x_3 &= -1 \\ x_2 - x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

Adding -3 times the second equation to the third equation eliminates the third equation.

Because the third equation is unnecessary, omit it to obtain the system shown below.

$$\begin{aligned} x_1 \qquad - 3x_3 &= -1 \\ x_2 - x_3 &= 0 \end{aligned}$$

To represent the solutions, choose x_3 to be the free variable and represent it by the parameter t. Because $x_2 = x_3$ and $x_1 = 3x_3 - 1$, you can describe the solution set as

$$x_1 = 3t - 1, \quad x_2 = t, \quad x_3 = t, \quad t \text{ is any real number.}$$

DISCOVERY

1. Graph the two lines represented by the system of equations.

$$\begin{aligned} x - 2y &= 1 \\ -2x + 3y &= -3 \end{aligned}$$

2. Use Gaussian elimination to solve this system as follows.

$$\begin{aligned} x - 2y &= 1 \\ -1y &= -1 \end{aligned}$$

$$\begin{aligned} x - 2y &= 1 \\ y &= 1 \end{aligned}$$

$$\begin{aligned} x &= 3 \\ y &= 1 \end{aligned}$$

Graph the system of equations you obtain at each step of this process. What do you observe about the lines?

You are asked to repeat this graphical analysis for other systems in Exercises 89 and 90.

1.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

Linear Equations In Exercises 1–6, determine whether the equation is linear in the variables x and y.

1. $2x - 3y = 4$
2. $3x - 4xy = 0$
3. $\frac{3}{y} + \frac{2}{x} - 1 = 0$
4. $x^2 + y^2 = 4$
5. $2 \sin x - y = 14$
6. $(\sin 2)x - y = 14$

Parametric Representation In Exercises 7–10, find a parametric representation of the solution set of the linear equation.

7. $2x - 4y = 0$
8. $3x - \frac{1}{2}y = 9$
9. $x + y + z = 1$
10. $13x_1 - 26x_2 + 39x_3 = 13$

Graphical Analysis In Exercises 11–24, graph the system of linear equations. Solve the system and interpret your answer.

11. $2x + y = 4$; $x - y = 2$
12. $x + 3y = 2$; $-x + 2y = 3$
13. $x - y = 1$; $-2x + 2y = 5$
14. $\frac{1}{2}x - \frac{1}{3}y = 1$; $-2x + \frac{4}{3}y = -4$
15. $3x - 5y = 7$; $2x + y = 9$
16. $-x + 3y = 17$; $4x + 3y = 7$
17. $2x - y = 5$; $5x - y = 11$
18. $x - 5y = 21$; $6x + 5y = 21$
19. $\frac{x+3}{4} + \frac{y-1}{3} = 1$; $2x - y = 12$
20. $\frac{x-1}{2} + \frac{y+2}{3} = 4$; $x - 2y = 5$
21. $0.05x - 0.03y = 0.07$; $0.07x + 0.02y = 0.16$
22. $0.2x - 0.5y = -27.8$; $0.3x + 0.4y = 68.7$
23. $\frac{x}{4} + \frac{y}{6} = 1$; $x - y = 3$
24. $\frac{2x}{3} + \frac{y}{6} = \frac{2}{3}$; $4x + y = 4$

Back-Substitution In Exercises 25–30, use back-substitution to solve the system.

25. $x_1 - x_2 = 2$; $x_2 = 3$
26. $2x_1 - 4x_2 = 6$; $3x_2 = 9$
27. $-x + y - z = 0$; $2y + z = 3$; $\frac{1}{2}z = 0$
28. $x - y = 4$; $2y + z = 6$; $3z = 6$
29. $5x_1 + 2x_2 + x_3 = 0$; $2x_1 + x_2 = 0$
30. $x_1 + x_2 + x_3 = 0$; $x_2 = 0$

Graphical Analysis In Exercises 31–36, complete the following for the system of equations.

(a) Use a graphing utility to graph the system.

(b) Use the graph to determine whether the system is consistent or inconsistent.

(c) If the system is consistent, approximate the solution.

(d) Solve the system algebraically.

(e) Compare the solution in part (d) with the approximation in part (c). What can you conclude?

31. $-3x - y = 3$; $6x + 2y = 1$
32. $4x - 5y = 3$; $-8x + 10y = 14$
33. $2x - 8y = 3$; $\frac{1}{2}x + y = 0$
34. $9x - 4y = 5$; $\frac{1}{2}x + \frac{1}{3}y = 0$
35. $4x - 8y = 9$; $0.8x - 1.6y = 1.8$
36. $-5.3x + 2.1y = 1.25$; $15.9x - 6.3y = -3.75$

System of Linear Equations In Exercises 37–56, solve the system of linear equations.

37. $x_1 - x_2 = 0$; $3x_1 - 2x_2 = -1$
38. $3x + 2y = 2$; $6x + 4y = 14$
39. $2u + v = 120$; $u + 2v = 120$
40. $x_1 - 2x_2 = 0$; $6x_1 + 2x_2 = 0$
41. $9x - 3y = -1$; $\frac{1}{5}x + \frac{2}{5}y = -\frac{1}{3}$
42. $\frac{2}{3}x_1 + \frac{1}{6}x_2 = 0$; $4x_1 + x_2 = 0$
43. $\frac{x-2}{4} + \frac{y-1}{3} = 2$; $x - 3y = 20$
44. $\frac{x_1+4}{3} + \frac{x_2+1}{2} = 1$; $3x_1 - x_2 = -2$
45. $0.02x_1 - 0.05x_2 = -0.19$; $0.03x_1 + 0.04x_2 = 0.52$
46. $0.05x_1 - 0.03x_2 = 0.21$; $0.07x_1 + 0.02x_2 = 0.17$
47. $x + y + z = 6$; $2x - y + z = 3$; $3x - z = 0$
48. $x + y + z = 2$; $-x + 3y + 2z = 8$; $4x + y = 4$
49. $3x_1 - 2x_2 + 4x_3 = 1$; $x_1 + x_2 - 2x_3 = 3$; $2x_1 - 3x_2 + 6x_3 = 8$
50. $5x_1 - 3x_2 + 2x_3 = 3$; $2x_1 + 4x_2 - x_3 = 7$; $x_1 - 11x_2 + 4x_3 = 3$

The symbol indicates an exercise in which you are instructed to use a graphing utility or a symbolic computer software program.

51. $$\begin{aligned} 2x_1 + x_2 - 3x_3 &= 4 \\ 4x_1 + 2x_3 &= 10 \\ -2x_1 + 3x_2 - 13x_3 &= -8 \end{aligned}$$

52. $$\begin{aligned} x_1 + 4x_3 &= 13 \\ 4x_1 - 2x_2 + x_3 &= 7 \\ 2x_1 - 2x_2 - 7x_3 &= -19 \end{aligned}$$

53. $$\begin{aligned} x - 3y + 2z &= 18 \\ 5x - 15y + 10z &= 18 \end{aligned}$$

54. $$\begin{aligned} x_1 - 2x_2 + 5x_3 &= 2 \\ 3x_1 + 2x_2 - x_3 &= -2 \end{aligned}$$

55. $$\begin{aligned} x + y + z + w &= 6 \\ 2x + 3y - w &= 0 \\ -3x + 4y + z + 2w &= 4 \\ x + 2y - z + w &= 0 \end{aligned}$$

56. $$\begin{aligned} x_1 + 3x_4 &= 4 \\ 2x_2 - x_3 - x_4 &= 0 \\ 3x_2 - 2x_4 &= 1 \\ 2x_1 - x_2 + 4x_3 &= 5 \end{aligned}$$

System of Linear Equations **In Exercises 57–60, use a software program or a graphing utility to solve the system of linear equations.**

57. $$\begin{aligned} x_1 + 0.5x_2 + 0.33x_3 + 0.25x_4 &= 1.1 \\ 0.5x_1 + 0.33x_2 + 0.25x_3 + 0.21x_4 &= 1.2 \\ 0.33x_1 + 0.25x_2 + 0.2x_3 + 0.17x_4 &= 1.3 \\ 0.25x_1 + 0.2x_2 + 0.17x_3 + 0.14x_4 &= 1.4 \end{aligned}$$

58. $$\begin{aligned} 120.2x + 62.4y - 36.5z &= 258.64 \\ 56.8x - 42.8y + 27.3z &= -71.44 \\ 88.1x + 72.5y - 28.5z &= 225.88 \end{aligned}$$

59. $$\begin{aligned} \tfrac{1}{2}x_1 - \tfrac{3}{7}x_2 + \tfrac{2}{9}x_3 &= \tfrac{349}{630} \\ \tfrac{2}{3}x_1 + \tfrac{4}{9}x_2 - \tfrac{2}{5}x_3 &= -\tfrac{19}{45} \\ \tfrac{4}{5}x_1 - \tfrac{1}{8}x_2 + \tfrac{4}{3}x_3 &= \tfrac{139}{150} \end{aligned}$$

60. $$\begin{aligned} \tfrac{1}{8}x - \tfrac{1}{7}y + \tfrac{1}{6}z - \tfrac{1}{5}w &= 1 \\ \tfrac{1}{7}x + \tfrac{1}{6}y - \tfrac{1}{5}z + \tfrac{1}{4}w &= 1 \\ \tfrac{1}{6}x - \tfrac{1}{5}y + \tfrac{1}{4}z - \tfrac{1}{3}w &= 1 \\ \tfrac{1}{5}x + \tfrac{1}{4}y - \tfrac{1}{3}z + \tfrac{1}{2}w &= 1 \end{aligned}$$

Number of Solutions **In Exercises 61–64, state why the system of equations must have at least one solution. Then solve the system and determine whether it has exactly one solution or infinitely many solutions.**

61. $$\begin{aligned} 4x + 3y + 17z &= 0 \\ 5x + 4y + 22z &= 0 \\ 4x + 2y + 19z &= 0 \end{aligned}$$

62. $$\begin{aligned} 2x + 3y &= 0 \\ 4x + 3y - z &= 0 \\ 8x + 3y + 3z &= 0 \end{aligned}$$

63. $$\begin{aligned} 5x + 5y - z &= 0 \\ 10x + 5y + 2z &= 0 \\ 5x + 15y - 9z &= 0 \end{aligned}$$

64. $$\begin{aligned} 12x + 5y + z &= 0 \\ 12x + 4y - z &= 0 \end{aligned}$$

65. Nutrition One eight-ounce glass of apple juice and one eight-ounce glass of orange juice contain a total of 177.4 milligrams of vitamin C. Two eight-ounce glasses of apple juice and three eight-ounce glasses of orange juice contain a total of 436.7 milligrams of vitamin C. How much vitamin C is in an eight-ounce glass of each type of juice?

66. Airplane Speed Two planes start from Los Angeles International Airport and fly in opposite directions. The second plane starts $\frac{1}{2}$ hour after the first plane, but its speed is 80 kilometers per hour faster. Find the airspeed of each plane if 2 hours after the first plane departs, the planes are 3200 kilometers apart.

True or False? **In Exercises 67 and 68, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.**

67. (a) A system of one linear equation in two variables is always consistent.

(b) A system of two linear equations in three variables is always consistent.

(c) If a linear system is consistent, then it has infinitely many solutions.

68. (a) A linear system can have exactly two solutions.

(b) Two systems of linear equations are equivalent when they have the same solution set.

(c) A system of three linear equations in two variables is always inconsistent.

69. Find a system of two equations in two variables, x_1 and x_2, that has the solution set given by the parametric representation $x_1 = t$ and $x_2 = 3t - 4$, where t is any real number. Then show that the solutions to the system can also be written as

$$x_1 = \frac{4}{3} + \frac{t}{3} \quad \text{and} \quad x_2 = t.$$

70. Find a system of two equations in three variables, x_1, x_2, and x_3, that has the solution set given by the parametric representation

$$x_1 = t, \quad x_2 = s, \quad \text{and} \quad x_3 = 3 + s - t$$

where s and t are any real numbers. Then show that the solutions to the system can also be written as

$$x_1 = 3 + s - t, \quad x_2 = s, \quad \text{and} \quad x_3 = t.$$

The symbol [icon] indicates that electronic data sets for these exercises are available at *www.cengagebrain.com*. These data sets are compatible with each of the following technologies: MATLAB, *Mathematica*, *Maple*, TI-83 Plus, TI-84 Plus, TI-89, Voyage 200.

Substitution **In Exercises 71–74, solve the system of equations by letting $A = 1/x$, $B = 1/y$, and $C = 1/z$.**

71. $\frac{12}{x} - \frac{12}{y} = 7$
$\frac{3}{x} + \frac{4}{y} = 0$

72. $\frac{2}{x} + \frac{3}{y} = 0$
$\frac{3}{x} - \frac{4}{y} = -\frac{25}{6}$

73. $\frac{2}{x} + \frac{1}{y} - \frac{3}{z} = 4$
$\frac{4}{x} + \frac{2}{z} = 10$
$-\frac{2}{x} + \frac{3}{y} - \frac{13}{z} = -8$

74. $\frac{2}{x} + \frac{1}{y} - \frac{2}{z} = 5$
$\frac{3}{x} - \frac{4}{y} = -1$
$\frac{2}{x} + \frac{1}{y} + \frac{3}{z} = 0$

Trigonometric Coefficients **In Exercises 75 and 76, solve the system of linear equations for x and y.**

75. $(\cos\theta)x + (\sin\theta)y = 1$
$(-\sin\theta)x + (\cos\theta)y = 0$

76. $(\cos\theta)x + (\sin\theta)y = 1$
$(-\sin\theta)x + (\cos\theta)y = 1$

Coefficient Design **In Exercises 77–82, determine the value(s) of k such that the system of linear equations has the indicated number of solutions.**

77. Infinitely many solutions
$4x + ky = 6$
$kx + y = -3$

78. Infinitely many solutions
$kx + y = 4$
$2x - 3y = -12$

79. Exactly one solution
$x + ky = 0$
$kx + y = 0$

80. No solution
$x + ky = 2$
$kx + y = 4$

81. No solution
$x + 2y + kz = 6$
$3x + 6y + 8z = 4$

82. Exactly one solution
$kx + 2ky + 3kz = 4k$
$x + y + z = 0$
$2x - y + z = 1$

83. Determine the values of k such that the system of linear equations does not have a unique solution.
$x + y + kz = 3$
$x + ky + z = 2$
$kx + y + z = 1$

84. CAPSTONE Find values of a, b, and c such that the system of linear equations has (a) exactly one solution, (b) infinitely many solutions, and (c) no solution. Explain your reasoning.
$x + 5y + z = 0$
$x + 6y - z = 0$
$2x + ay + bz = c$

85. Writing Consider the system of linear equations in x and y.
$a_1x + b_1y = c_1$
$a_2x + b_2y = c_2$
$a_3x + b_3y = c_3$
Describe the graphs of these three equations in the xy-plane when the system has (a) exactly one solution, (b) infinitely many solutions, and (c) no solution.

86. Writing Explain why the system of linear equations in Exercise 85 must be consistent when the constant terms c_1, c_2, and c_3 are all zero.

87. Show that if $ax^2 + bx + c = 0$ for all x, then $a = b = c = 0$.

88. Consider the system of linear equations in x and y.
$ax + by = e$
$cx + dy = f$
Under what conditions will the system have exactly one solution?

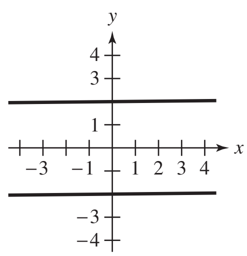
Discovery **In Exercises 89 and 90, sketch the lines represented by the system of equations. Then use Gaussian elimination to solve the system. At each step of the elimination process, sketch the corresponding lines. What do you observe about the lines?**

89. $x - 4y = -3$
$5x - 6y = 13$

90. $2x - 3y = 7$
$-4x + 6y = -14$

Writing **In Exercises 91 and 92, the graphs of the two equations appear to be parallel. Solve the system of equations algebraically. Explain why the graphs are misleading.**

91. $100y - x = 200$
$99y - x = -198$



92. $21x - 20y = 0$
$13x - 12y = 120$